§4. SOLUTIONS TO MAXWELL'S EQUATIONS

The equations given in Eq. (3.69) are almost perfectly symmetric in **E** and **B**. The solution to these four equations will be done using two different approaches. The first will develop the wave equation by using vector algebra. The second approach will make use of the vector and scalar potentials. Both methods result in Maxwell's wave equations, however the vector potential solution will prepare the way for the quantum mechanical description of the propagating electromagnetic wave.

§4.1. VECTOR ALGEBRA SOLUTION TO MAXWELL'S EQUATIONS

If the two Divergence equations are set aside for a moment, the two Curl equations describe coupled electric and magnetic fields since they both contain **E** and **B**. This approach to the solution will eliminate the duplicate terms and produce the wave equation directly.

Taking the Curl of Eq. (3.69b) gives,

$$\nabla\times(\nabla\times\mathbf{E}) = -\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) \quad (4.1)$$

inserting Eq. (3.69d) into Eq. (4.1) gives,

$$\nabla\times(\nabla\times\mathbf{E}) \equiv \frac{\partial}{\partial t}\left(-\frac{\partial\mathbf{E}}{\partial t}\right) = -\frac{\partial^2\mathbf{E}}{\partial t^2}. \quad (4.2)$$

Since $\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - (\nabla\cdot\nabla)\mathbf{E}$, the first term of Eq. (4.2) is zero, because Gauss's law states $\nabla\cdot\mathbf{E} = 0$ and the second term is the definition of the Laplacian operating on **E**. In Cartesian coordinates,

$$\nabla^2\mathbf{E} \equiv \frac{\partial^2\mathbf{E}}{\partial x^2} + \frac{\partial^2\mathbf{E}}{\partial y^2} + \frac{\partial^2\mathbf{E}}{\partial z^2} = \frac{\partial^2\mathbf{E}}{\partial t^2} \quad (4.3)$$

The same manipulation can take place for Eq. (3.69c) giving,

$$\nabla^2\mathbf{B} = \frac{\partial^2\mathbf{B}}{\partial t^2} \quad (4.4)$$

The results of Eq. (4.3) and Eq. (4.4) are Maxwell's wave equations propagating through free space.

§4.2. VECTOR POTENTIAL SOLUTION TO MAXWELL'S EQUATIONS

Maxwell's equations can be solved as they stand in simple situations, but it is often convenient to introduce potentials, obtaining a smaller number of second–order equations while satisfying some of the Maxwell equations with an identity. [1]

Since $\nabla \cdot \mathbf{B} = 0$ (Eq. (II)) still holds, **B** can be defined in terms of a vector potential, such that,

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{4.5}$$

Faraday's Law Eq. (IV) can be written as,

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 . \tag{4.6}$$

This means that the quantity with a vanishing curl in Eq. (4.6) can be written as the gradient of some scalar function,

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi , \tag{4.7}$$

or,

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} . \tag{4.8}$$

The definitions of **B** and **E** in terms of the potentials **A** and ϕ will be determined by the homogeneous equations Eq. (3.8) and Eq. (3.30) [2] The

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[1] The solutions to Maxwell's equations using the scalar and vector potential is the *modern* approach and is given in hindsight as the *logical* approach to the problem. As is usual with such revisionist matters the logic of this approach was not obvious to the investigators of Maxwell's time, but the developed over a period of intense effort, resulting in the hindsight of today [Buch85].

[2] The origins of the potential solution to Maxwell's equations is obscured by history. The earliest accounts of *partitioning* any well–behaved vector field into its irrotational (curl free) and solenoidial components can be found in [Helm58]. In general, vector fields are determined by the knowledge of their curl and divergence.

In 1846 — the year after he had taken his degree as second wrangler at Cambridge, J. J. Thompson (1856–1940) investigated the analogy between electric phenomena and mechanical elasticity. He examined the equations of equilibrium of an incompressible elastic solid which is under strain. He showed that the distribution of the vector which represents the displacement could be compared to the distribution of the electric force in a electrostatic system. The elastic displacement could be identified with a vector **A**, defined in inhomogeneous equations, Eq. (3.5) and Eq. (3.24), can be written in terms of the potentials as,

$$\nabla^2\phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\rho , \tag{4.9}$$

and,

$$\nabla^2\mathbf{A} - \frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{\partial\phi}{\partial t}\right) = -\mathbf{j}. \tag{4.10}$$

The four Maxwell equations are now reduced to two coupled inhomogeneous differential equations, Eq. (4.9) and Eq. (4.10) The uncoupling of these equations can be accomplished by exploiting the arbitrariness involved in the definition of the potentials [Brom70]. Since the **B** field is defined through Eq. (68) in terms of **A**, the vector potential is arbitrary to the extent that the gradient of some scalar function χ can be added to the vector potential. [3] The magnetic field **B** is left unchanged by the transformation,

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\chi . \tag{4.11}$$

In order that the electric field be unchanged as well, the scalar potential must be simultaneously transformed as,

$$\phi \to \phi' = \phi - \frac{\partial\chi}{\partial t}. \tag{4.12}$$

The transformations in equation Eq. (4.11) and Eq. (4.12) are called *gauge transformations* and the invariance a *gauge invariance*.

The solution to Maxwell's equations using the scalar potential field ϕ and vector potential field **A**, which in turn relate to the electric and magnetic fields by Eq. (3.5) and Eq. (3.24) [Eyge72] can be further

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terms of the magnetic induction **B**, by the familiar $\nabla \times \mathbf{A} = \mathbf{B}$. The vector **A** is equivalent to the vector potential which had been mentioned in the memoirs of Weber and Kerchief on the induction of currents; but Thompson arrived at this independently. Although Thompson laid the groundwork, it was J. C. Maxwell who provided the solution to the electromagnetic propagation question through his wave equation formulation.

[3] The vector **A** is not completely determined by the magnetic field **B**, since for any scalar function χ, $\nabla \times \nabla\chi = 0$ the gradient of an arbitrary function χ can be added to the vector field **A**. For the scalar potential ϕ, the time derivative of the arbitrary scalar function χ is subtracted in order to maintain the electric field's invariance.

simplified using several techniques. [4] The solutions provided below are given without the formalism of their derivations which can be found in [Feyn64], [Cowa68] and [Land71]. One approach to the solution of Maxwell's equations is the use of Green's Theorem, developed in detail in the next sections. The approach is taken with the intention of developing a *radiation* oriented set of equations that can be used to solve problems in the *radiation zone* of the electromagnetic field.

By using the vector identity, $\nabla\times\nabla\times\mathbf{A}=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$, and substituting Eq. (4.1) and Eq. (4.4) into Maxwell's equations, results in the corresponding scalar and vector field equations,

$$\left.\begin{aligned}
\nabla\cdot\mathbf{E}=\rho \quad &\Rightarrow \quad \nabla^2\phi-\frac{\partial^2\phi}{\partial t^2}+\frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{A}+\frac{\partial\phi}{\partial t}\right)=-\rho, && (a)\\
\nabla\cdot\mathbf{B}=0 \quad &\Rightarrow \quad \nabla\cdot\left(\nabla\times\mathbf{A}\right)=0., && (b)\\
\nabla\times\mathbf{B}=\mathbf{j}+\frac{\partial\mathbf{E}}{\partial t} \quad &\Rightarrow \quad \nabla^2\mathbf{A}-\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla\left(\nabla\cdot\mathbf{A}+\frac{\partial\phi}{\partial t}\right)=-\mathbf{j}, && (c)\\
\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \quad &\Rightarrow \quad \nabla\times\left(-\frac{\partial\mathbf{A}}{\partial t}-\nabla\phi\right)=-\frac{\partial}{\partial t}\nabla\times\mathbf{A} && (d)
\end{aligned}\right\} \tag{4.13}$$

§4.3. INTEGRAL FORM OF MAXWELL'S FIELD EQUATIONS

The freedom of defining an arbitrary scalar and vector potential implied by Eq. (4.11) and Eq. (4.12) means that a set of potentials can be chosen such that,

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[4] The introduction of the scalar and vector potential fields is motivated by the search for solutions to the $\mathbf{E}$ and $\mathbf{B}$ fields whose form is appropriate for the traveling waves development in the next section. The vector and scalar potentials in Eq. (4.7)) and Eq. (4.8) are not unique. The simplification obtained by the introduction of $\mathbf{A}$ and ϕ must be paid for by the fact that ϕ and $\mathbf{A}$ are not unique for a given $\mathbf{E}$ and $\mathbf{B}$ field. Starting with a given choice of $\mathbf{A}$ and ϕ, the same fields may be obtained from the alternative potentials, $\mathbf{A}\to\mathbf{A}'=\mathbf{A}+\nabla\chi$ and $\phi\to\phi'=\phi-\partial\chi/\partial t$ where χ is an arbitrary scalar function that does not effect the individual field components. Since it is the fields which are the observable quantities there is no physical basis for choosing $\mathbf{A}$ or $\mathbf{A}'$. The field calculated from $\mathbf{A}'$ is the same as the one calculated from $\mathbf{A}$. Therefore only those quantities *invariant* under the gauge transformation will have direct experimental significance. The transformation is called a *gauge transformation* and is further developed in the section on *Gauge Theory*; the Maxwell equations are said to be invariant with respect to gauge transformations. The useful gauge transformations found in electrodynamics are: the *Lorentz Gauge*: $\partial\mathbf{A}/\partial x_i=0$, the *Radiation Gauge*: $\nabla\cdot\mathbf{A}=0$ and the *Coulomb Gauge*: $\nabla\cdot\mathbf{A}=0,\phi=0$. The concept of gauge invariance and its relationship to the electromagnetic potential is developed further in later sections.

$$\nabla \cdot \mathbf{A} + \frac{\partial \phi}{\partial t} = 0, \tag{4.14}$$

that defines the Lorentz condition which allows an arbitrary specification of the vector and scalar potentials. [5] The Lorentz condition can now be used to uncouple the pair of equations Eq. (4.9) and Eq. (4.10) and leave two inhomogeneous equations [Borm70] (which later will be developed as the wave equations) one for ϕ and one for $\mathbf{A}$,

$$\left.\begin{aligned} \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} &= -\rho, \\ \nabla^2 \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mathbf{j}. \end{aligned}\right\}. \tag{4.15}$$

§4.3.1. Green's Function and the Potential Solution

The problem[6] of finding the solution to the potential equations $\mathbf{A}$ and ϕ in terms of currents and charges may be approached using Fourier analysis and its differential equation solution technique, Green's Functions. Using Green's functions [Gree71], [Boch01], [Butk68], [Mors53], [Arfk85], and the Lorentz gauge is an approach that allows the introduction of *gauge invariance* — which will later be important in the quantum mechanical description of the radiation field. [7] Using Eq. (4.13)

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[5] The Lorentz condition is not as arbitrary as some texts state, in that it leads to a symmetry between the vector and scalar potentials that allows these potentials to satisfy the same wave equation. In addition the Lorentz condition also provides a relativistic covariant relation between the scalar and vector potentials [Pano66].

[6] This section is one the *diversions* necessary to explain what is usually glossed over in many text books. The solution to Maxwell's equations using the scalar and vector potential makes use of the Green's function method. In this method partial differential equations may be solved in a straight forward manner, without explicit consideration of the boundary conditions.

[7] This theorem was first presented in 1828 by Georg Green (1793–1841) in his "Essay on the Application of Mathematics to Electricity and Magnetism" [Maxw65]. In the general theory of boundary value problems, an important role is played by a mathematical theorem called *Green's Theorem* and by certain integral expressions for the potential that are derived from it. Consider a volume V bounded by a closed surface S, let $\mathbf{n}$ be an outwardly pointing normal to the surface. Let $\mathbf{r}'$ be a position vector in a coordinate system with arbitrary origin. Given two arbitrary functions $\psi(\mathbf{r}')$ and $\chi(\mathbf{r}')$ that are appropriately continuous in V and form the function $\mathbf{A}(\mathbf{r}') = \psi(\mathbf{r}')\nabla\chi(\mathbf{r}')$. Then the divergence of the

(a) and Eq. (4.13)(b), a Green's function will be used to construct a general solution of these potential equations in integral form. A Green function $G(\mathbf{r}, t; \mathbf{r}', t')$, where $\mathbf{r}'$ and t' are passive parameters that satisfy,

$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -4\pi\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{4.16}$$

will be used as the basis of the solution of Maxwell's potential equations.

Beginning with the point source charge at the origin $\mathbf{r}' = t' = 0$, $G_0(\mathbf{r}, t) = G(\mathbf{r}, t; \mathbf{0}, 0)$ such that G_0 satisfies the Laplacian,

$$\nabla^2 G_0 - \frac{1}{c^2}\frac{\partial^2 G_0}{\partial t^2} = -4\pi\delta(\mathbf{r})\delta(t). \tag{4.17}$$

Since the source is a point charge, the solution is dependent only on $r = |\mathbf{r}|$.

Using the Laplacian in spherical coordinates, but with the angular coordinate, ϕ, equal to zero, Eq. (4.17) becomes,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G_0}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2 G_0}{\partial t^2} = -4\pi\delta(\mathbf{r})\delta(t). \tag{4.18}$$

The solution to this equation at a distance from the origin is,

$$G_0(\mathbf{r}, t) = \frac{f(t \pm r/c)}{r}, \tag{4.19}$$

where f is an arbitrary function of time and space.

Integrating Eq. (4.18) over a small volume ΔV containing the origin gives,

$$\int_{\Delta V} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G_0}{\partial r}\right)dv - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_{\Delta V} G_0 dv = -4\pi\delta(t) \tag{4.20}$$

---

function is given by $\nabla \cdot \mathbf{A} = \nabla\psi \cdot \nabla\chi + \psi\nabla^2\chi$. The divergence theorem $\int_V \nabla \cdot \mathbf{A} dv' = \int_S \mathbf{A} \cdot ds$ yields $\int_V (\nabla\psi \cdot \nabla\chi + \psi\nabla^2\chi) dv' = \int_S \psi\nabla\chi \cdot ds$. Writing a similar equation with the roles of ψ and χ interchanged and subtracting the two equations and using $(\psi\nabla\chi) \cdot ds' = \psi(\partial\chi/\partial n) ds'$ gives $\int_V (\psi\nabla^2\chi + \chi\nabla^2\psi) dv' = \int_S [\psi(\partial\chi/\partial n) - \chi(\partial\psi/\partial n)] ds'$. This relation between the surface and volume integrals is *Green's Theorem*.

With G_0 of Eq. (4.19), the integrand of the first integral in Eq. (4.20) is singular resulting in an improper integral.

The integral in Eq. (4.20) can be properly redefined by recognizing the integrand is $\nabla^2 G_0 = \nabla \cdot \nabla G_0$. By the divergence theorem the volume integral of the divergence can be converted to a surface integral over the small bounding surface $\Delta S \to 0$. [8]

With the conversion from a volume integral to a surface integral, the δ function condition in Eq. (4.20) becomes,

$$\int_{\Delta S} (\nabla G_0) \cdot ds - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int_{\Delta V} G_0 dv = -4\pi\delta(t). \tag{4.21}$$

Substituting the explicit form Eq. (4.19) into the equation and using,

$$\nabla G_0 = \left(\frac{f}{r^2} \pm \frac{1}{c} \frac{f'}{r} \right) \cdot \mathbf{r}, \tag{4.22}$$

where the prime denotes differentiation with respect to the argument of the function and r_0 is the radius of the small spherical volume about the origin, gives,

$$\left(-\frac{f\left[t \pm r_0/c\right]}{r_0^2} \mp \frac{f'\left[t \pm r_0/c\right]}{c r_0} \right) 4\pi r_0^2 - \frac{1}{c^2} \int \frac{\delta''}{r} dv = -4\pi\delta(t). \tag{4.23}$$

Letting $r_0 \to 0$ results in $f(t) = \delta(t)$, giving,

$$G_0(r, t) = \frac{\delta(t - t' \pm |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|}, \tag{4.24}$$

and,

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta(t - t' \pm |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|}. \tag{4.25}$$

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[8] The *divergence theorem* states that for any well–behaved vector field $\mathbf{A}(x)$ defined within a volume V surrounded by the closed surface S the relation $\oint A \cdot n da = \int_V \nabla \cdot A dv$ holds between the volume integral of the outwardly directed normal component of $\mathbf{A}$. This relation can be used as the definition of the *divergence* of a vector field [Stra41], pg. 4.

§4.3.2. Field Potential Solutions

This Green function allows the integration of Eq. (4.13)(a) and Eq. (4.13)(b) so that the solution to ϕ and $\mathbf{A}$ is,

$$\phi(\mathbf{r},t)=\int\frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}dv, \tag{4.26}$$

$$\mathbf{A}(\mathbf{r},t)=\int\frac{\mathbf{j}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}dv. \tag{4.27}$$

The solutions provided by equations Eq. (4.26) and Eq. (4.27) are particular integrals of the inhomogeneous equations Eq. (4.13)(a) and Eq. (4.13)(b). For the purpose of adapting to given initial and boundary conditions, integrals of the homogeneous potential equations can be added to produce the wave equations for $\mathbf{A}$ and ϕ. [9]

In Eq. (4.26) and Eq. (4.27), $t-|\mathbf{r}-\mathbf{r}'|/c$ denotes, that for $\phi(t)$, the value of the charge density ρ at time $t-|\mathbf{r}-\mathbf{r}'|/c$ should be used. That is, for each element of charge ρdv, the equation states that the contribution to the potential is the same form as in the static charge density equation, $\phi=\frac{1}{4\pi r}\int\rho dV$, except that the finite propagation time for the charge effect must be accounted for. For computing the total contribution to the potential ϕ at a point x at time t, the values of charge density from points distance $\mathbf{r}'$ away at an earlier time $t-|\mathbf{r}-\mathbf{r}'|/c$, since for a given element it is that effect which reaches x at time t. A similar interpretation applies to the computation of $\mathbf{A}$ from currents in Eq. (4.27). Because of this *retardation* effect, the potentials $\phi(t)$ and $\mathbf{A}$(t) are called the *retarded potentials*. [10]

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[9] The solutions to the inhomogeneous Maxwell equations using Green's functions is based on the existence of the Fourier transforms of the vector and scalar potential functions. These solutions do not in principal apply to monochromatic radiation sources. Using the Fourier transform, a monochromatic source (radiating at a single frequency) would radiate over an infinite time. This situation can be delete with if a *limiting* process is used starting with a finite duration time domain pulse.

[10] The physical content of equations Eq. (4.22) and Eq. (4.23) is not identical with that of Eq. (4.11) and Eq. (4.11) While in the differential form the *sign of the time* is in no way distinguished, i.e. the equations are not altered by an exchange of past with future, the integral forms make an essential distinction between past and future. Mathematically,

Maxwell's equations (I) – (V) provide several relations between the vector potential **A** and the scalar potential ϕ. By virtue of Maxwell's Equation (I),

$$\nabla \cdot \mathbf{A} = 0, \quad (4.28)$$ [11]

and,

$$\nabla^2 \phi = -\rho. \quad (4.29)$$ [12]

As the electromagnetic wave propagates through sdpace, energy is transferred form the source (transmitter antenna) to the destination (receiving antenna). This energy is subject to the Laws of Conservation of Energy given by,

$$\frac{d}{dt}\int_V U dV = -\int_S S \cdot da \quad (4.30)$$

The energy density U is given by the instantaneous value of the electric and magnetic fields,

(VI) $$U_{\mathbf{E}} = \frac{1}{2}\mathbf{E}^2 \text{ joules/m}^3, \quad (4.31)$$

and,

(VII) $$U_{\mathbf{B}} = \frac{1}{2}\mathbf{B}^2 \text{ joules/m}^3. \quad (4.32)$$

Eq. (4.22) and Eq. (4.23) would also be possible in which values of $\mathbf{j}$ and ρ at the source point are chosen for later time, $t + \mathbf{r}/c$, giving the *advanced potentials*. Such solutions would, however be contradictory to the concepts of causality, since charges and currents are considered to by the sources of potentials, since the electromagnetic field does not proceed the charges and currents which cause it [Pano66].

A theory of radiation involving the *advanced* potential was put forward by J. A. Wheeler (1911–) and R. P. Feynman (1918–1988) [Whee45] in which a covariant *action–at–a–distance* theory of electrodynamics can be formulated in terms of the symmetrical combination of the retarded and advanced potentials [Pano66].

[11] This is the result of the choice of the *Lorentz Gauge*.

[12] This is Poisson's equation and it is the description of an irrotational vector field derived from its sources. The solution may be found using Green's theorem [Gree71] in which ϕ is equal to the desired potential function and ψ is set to $1/r$ giving, $\oint\{(1/r)(\partial\phi/\partial N) - \phi(\partial 1/r/\partial N)\}\,ds = \int\{(1/r)\nabla^2\phi - \phi\nabla^2(1/r)\}\,dV$.

where the total energy density is given by,

$$U = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) \tag{4.33}$$

Eq. (4.31) and Eq. (4.32) represent the electrical and magnetic energy densities of the *microscopic* electromagnetic field. These energies reside *in the field* itself in a localized volume element.

This volume element contains the total energy of,

$$\frac{1}{2}\int_V \left(\mathbf{E}^2 + \mathbf{B}^2\right) d^3x \,. \tag{4.34}$$

This result form the basis of the Poynting Theorem developed in §5.2 and the Lagrangian and Hamiltonian approach to quantizing the electromagnetic field.

Finally Maxwell's Equation (III) and (I) imply that charge is conserved through an equation of continuity,

(VIII) $$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \tag{4.35}$$

§4.4. TRAVELING WAVES

The four Maxwell equations, the continuity equation and the energy density equation, represent the sum of all knowledge regarding classical electrodynamics in the early twentieth century. From these equations all macro–world physics can be derived, since they describe the interaction of electromagnetism, including light, and matter in non–quantum mechanical and non–relativistic terms.

> *Life is a wave, which in no two consecutive moments of its existence is composed of the same particles.*
>
> — John Tyndall

The mathematical form of Maxwell's equations ((I) – (IV)) leads to the *discovery* of wave–like motion of the electric and magnetic fields. [13] The

[13] Heinrich Rudolf Hertz (1857–94) devised an experimental test of Maxwell's *traveling wave* theory [Suss64], [D'ago75], [Hert95], [Long83], [Buch94]. He constructed a *spark–gap* generator which was used as a transmitter and a loop of wire as a receiver. The spark produced by the transmitter would produce a similar spark between a gap in the receiving loop. Using a zinc plate Hertz showed the *standing waves* were present. By moving the

equations also indicate that the speed of the waves should be a constant, related to the Permittivity and Permeability of free space. When Maxwell calculated the value of this constant, he found it ...

> … so nearly that of light, that it seems we have strong reason to conclude that light itself ... is an electromagnetic disturbance in the form of waves propagating through the electromagnetic field according to the electromagnetic laws [Zaja74].

The existence of such waves was known theoretically prior to the engineering skills necessary to construct equipment capable utilizing them — *a feat of theoretical physics not recently repeated.* [14]

To this point the equations describing the electric and magnetic fields have assumed that the waves are propagating in free space, free of any sources of the electromagnetic radiation, e.g. $\rho = \mathbf{j} = 0$. The solutions to Maxwell's equations ((I) – (IV)) describe electromagnetic plane waves that are transverse to the direction of propagation, given by the vector k, such that the electric and magnetic field vectors are mutually orthogonal with k. [15]

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receiving antenna, the intensity of the *received* spark would vary. He confirmed these *electric waves* would pass through wooden doors, be reflected like light and were polarized. Faraday's *lines of force* as well as Maxwell's *electric waves* were confirmed leading to the electromagnetic devices of today.

14 The electromagnetic waves described by Maxwell's equations are classified into several types, although they are all part of a continuous spectrum whose wave lengths range from $10^{2}m$ for *radio waves* to $10^{-2}m$ for microwaves, $10^{-10}m$ for x–rays through $10^{-15}m$ for cosmic rays. Maxwell predicted the existence of electromagnetic waves on theoretical grounds when he derived the *wave equation*.

The velocity of these *waves* depends n certain electrical constants — the permittivity and permeability of the propagating medium. The measurement of these constants along with Michael Faraday's experimental discovery that polarized light is rotated in the presence of a magnetic field led Maxwell to speculate that *light* was also an electromagnetic wave. Direct evidence of Maxwell's prediction came in 1888 in experiments performed by Heinrich Hertz (1857–1894) [Buch94].

15 The existence of *transverse* modes in the propagation of electromagnetic waves was first proposed by Augustin Jean Fresenel (1788–1827). In 1814 Fresenel wrote that he suspected light and heat were connected with the vibrations of a fluid. His concept that light was a form of motion of a medium was basic to his theory of optics. In 1821 he had reformulated his theory of optics in terms of *waves propagating in a medium* [Harm82]. Fresenel submitted a paper to the Paris Academy prize in 1819 which described the mathematical theory of the interference of light. His theory was confirmed by Thomas Young (1773–1829) through Young's *double slit* experiment. By 1821 Fresenel had formulated a theory of the polarization of light and realized that the vibrations of the

In order to develop the description of the refraction of electromagnetic waves in a later section, the influence of a material medium on the propagation of the electromagnetic waves must be addressed. Maxwell proposed that a *dielectric* constant ε be added to the wave equations such that the propagation velocity is given by $v = 1/\varepsilon$. This *simple* model can be used to describe wave propagation in an isotropic, non–absorbant dielectric, homogeneous medium.

In modern notation, the *Traveling Wave* equation can be derived from Maxwell's equations. By noting that Maxwell's equations are functions of time, which implies the $\mathbf{E}$ and $\mathbf{B}$ fields are not independent, all four of Maxwell's equations are needed for their solution. The two divergence equations (I) and (II) state that the flux of $\mathbf{E}$ and $\mathbf{B}$ outward through any volume in free space (in the absence of any charges) is zero. The two curl equations (III) and (IV) require that the $\mathbf{E}$ and $\mathbf{B}$ fields are coupled and imply if $\partial\mathbf{B}/\partial t$ is non zero, then so is $\nabla\times\mathbf{E}$. The curl of the $\mathbf{E}$ field can only be non zero if $\mathbf{E}$ is a function of position. If $\mathbf{B}$ is a function of time ($\partial\mathbf{B}/\partial t$) then $\nabla\times\mathbf{E}$ is also a function of time. Eq. (IV) now states that a $\mathbf{B}$ field which varies with time *generates* an $\mathbf{E}$ field which varies in both time and space.

In a similar manner a non zero $\partial\mathbf{E}/\partial t$ *generates* a time and space varying $\mathbf{B}$ field. The *coupling* of these two fields forms the basis of electromagnetic wave propagation.

The equation relating the spatial variations of the $\mathbf{E}$ field to the time variations of the $\mathbf{B}$ field can be obtained by eliminating $\mathbf{B}$ from the two curl equations (III) and (IV). This is done by taking the curl of Eq. (IV), $\nabla\times(\nabla\times\mathbf{E}) = \partial/\partial t\,(\nabla\times\mathbf{B})$, using the vector identity $\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E}$ and substituting into Eq. (3.30) to produce,

$$-\nabla^2\mathbf{E} + \nabla\times\frac{\partial\mathbf{B}}{\partial t} = 0. \tag{4.36}$$

In Eq. (4.36) the ∇ and the $\partial/\partial t$ operations can be interchanged, so that using Eq. (3.24) results in the electric field wave equation,

medium must be *purely transverse*. If this medium was composed of *molecules bound by forces acting at a distance* than transverse waves could not be propagated since the *aether* would have to be rigid. The problem of constructing a model of the aether that could propagate transverse rather than longitudinal waves became a major problem of nineteenth century optical physics [Harm82].

$$\nabla^2\mathbf{E} - \frac{\partial^2\mathbf{E}}{\partial t^2} = 0. \tag{4.37}$$

Similarly, taking the curl of Eq. (3.24) and using Eq. (3.8) and Eq. (3.24) gives the magnetic field wave equation,

$$\nabla^2\mathbf{B} - \frac{\partial^2\mathbf{B}}{\partial t^2} = 0. \tag{4.38}$$

Both vectors for **E** and **B** satisfy the same differential equation and describe coupled electric and magnetic fields propagating through space at the speed of light. [16]

§4.4.1. Displacement Current in the Field Equations

In §3.4 the displacement current term in Maxwell's equation,

$$\nabla\times\mathbf{B} = \mathbf{j} + \frac{\partial\mathbf{E}}{\partial t}, \tag{4.39}$$

describes the displacement of the physical media carrying the electromagnetic force. In the theory of continuous media, such as a material –– either a conducting media or a dielectric.

In free space, the measuring of the displacement current is less clear, since it involves the flow of a current which is caused by the flow of charge somehow connected by the displacement of the media. In Maxwell's original formulation of the propagation of electromagnetic waves, a mechanical oscillation was *visualized* which carried the wave. The displacement of this media could account for the displacement current.

[16] A wave is described as *plane homogeneous* when it is possible to place a family of parallel planes so that along each one of these planes the magnetic field strength does not change. Since **E** and **B** are constant along wave planes, all partial derivatives with respect to z and y vanish. The x–component of the two curl equations and the two divergence equations read $\partial\mathbf{E}_x/\partial t = \partial\mathbf{B}_x/\partial t = 0$ and $\partial\mathbf{E}_x/\partial x = \partial\mathbf{B}_x/\partial x = 0$ while the remaining components of the curl equations read $-\partial\mathbf{B}_z/\partial x = -\partial\mathbf{E}_y/\partial t$, $\partial\mathbf{B}_y/\partial x = \partial\mathbf{E}_z/\partial t$, $\partial\mathbf{E}_y/\partial t = -\partial\mathbf{B}_z/\partial t$ and $-\partial\mathbf{E}_z/\partial x = -\partial\mathbf{B}_y/\partial t$. These developments led Maxwell to conclude that the traveling waves of his electromagnetic theory behaved like the *transverse* waves of the previously observed light waves.

This discovery was hailed as a dramatic confirmation of Maxwell's theoretical description of electromagnetism. Within a decade Marconi and others were using *Hertzian* waves in practical applications. Unfortunately Hertz did not live to see these deices. He died of blood poisoning at age 36 in 1894 [Buch94].

Once the aether was removed, the displacement current is no longer *visible*. By using the *wave equations* the displacement term of Eq. (4.39) can be eliminated.

§4.5. CLASSICAL EXPLANATIONS FOR FORCE FROM FIELDS

Now that the classical theory of fields has been developed and the resulting wave equation derived — what is force that is actually felt by the particle imbedded in the electromagnetic field at a distance from the accelerating charge? By what mechanism is this force carried to the charged particle?

Using the Lorentz force function,

$$\mathbf{F} = q\left[\mathbf{E} + \left(v \times \mathbf{B}\right)\right], \tag{4.40}$$

expressed in terms of the scalar potential ϕ and the vector potential $\mathbf{A}$, the electric and magnetic field become,

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{4.41}$$

and,

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{4.42}$$

Since these equations do not uniquely specify ϕ and $\mathbf{A}$, Maxwell's equations take their simplest form when the scalar and vector potentials are related through the Lorentz condition,

$$\nabla \cdot \mathbf{A} + \frac{\partial \phi}{\partial t} = 0, \tag{4.43}$$

which gives the Lorentz force as,

$$\mathbf{F} = q\left[-\nabla\left(\phi - v \cdot \mathbf{A}\right) - \frac{d\mathbf{A}}{dt}\right]. \tag{4.44}$$

Maxwell's equations permit, in principle at least, the calculation of the fields $\mathbf{E}$ and $\mathbf{B}$ from arbitrary sources. Since these fields are important because of their actions on charges, the foundations of electromagnetic theory are completed by a description of the Lorentz Force density $\mathbf{F}$ as shown in Eq. (4.44). It should be remembered that Eq. (4.44) is a postulate, but it is illuminating to see its origin [Whit51]. The first terms in Eq. (4.44) extends the definition of $\mathbf{E}$, as the force exerted on a unit charge, to a force exerted on a charge by a time varying field. The second term is the foundation of the postulate – it generalizes the magnetostatic results on the force between stationary currents circulating in loops of wire. Lorentz assumed that the current in a wire was due to the motion of individual, microscopic, charged particles. Formally this assumption is given by the current as $I = dq/dt$, with the current in a loop of wire given as $Idl = dq\, dl/dt$. Interpreting dl/dt as the velocity v of the charge dq, the force $d\mathbf{F}$ on this charge dq in motion is $d\mathbf{F} = dq\dfrac{v \times \mathbf{B}}{c}$, from the Biot–Savart law, which gives the origin of Eq. (4.44). Although the force density formula was inspired by the results of experiments on ensembles of charges that made up stationary currents, has been confirmed for general distributions of charges in arbitrary motion.

§4.6. Summary of Classical Field Theory

This classical *rationalization* of force derived from potential fields provides the explanation for the observed effects of the electromagnetic force. The Lorentz force law describe in Eq. (4.44), plus measurements of the components of acceleration of the test particle, can be viewed as defining the components of the electric and magnetic field. Once the field components are known from the accelerations of a test particle, they can be used to predict the accelerations of other test particles. The Lorentz force law is both the definer of fields and a predictor of particle motions.

Maxwell developed his ideas is a series of papers between 1861 and 1868. Subsequent experimental and theoretical investigations demonstrated a remarkable range of applicability of the theory [Buch85]. Maxwell's equations encompass light waves and the phenomena of optics; they turn out to be consistent with Einstein's special relativity [17] — in 1927 they were put in quantum form by P. A. M. Dirac [Dira27].

[17] Although revisionist history has placed Albert Einstein's accomplishments in light of his failure to *unify* gravity and electromagnetism his work during the year of 1905 was *breathtaking*. During 1905 when Einstein was 26 he published his first great work — a paper describing the theory of the photoelectric effect. It was in this paper he formulated the concept that light consists of *quanta* or photons. In the same year he published the theory of Brownian (Robert Brown (1773–1858)) motion — the movement of fine particles in a liquid — which laid the groundwork for the field of statistical mechanics. A third paper on the special theory of relativity was followed by a fourth paper in which he derived the most *popular* expression in modern science $E = mc^2$.

What remains — is the explanation for the cause of this force.

The search for this explanation leads to the next level of physical theory, developed at the beginning of the twentieth century — Quantum Mechanics. Before developing the concepts of quantum mechanics a the effects electromagnetic force on a remote charged particle will be examined. This will require a description of the radiated field and antenna theory. This will be the subject of the next section.

Before proceeding with the next section a short summary of the progress made so far is useful. Electrostatics and electromagnetic can be described using Maxwell's equations. From the original four equations *electromagnetic waves* were *deduced* which led to the engineering field of radio transmissions.

In the next section, Maxwell's equations will be used to define:

- The energy contained in the electromagnetic field. This field energy will be used to provide the force necessary to move the electrons in the remote antenna.
- The vector and scalar potential fields will be defined. It will be through these new fields that the electromagnetic field will be *quantized* in later sections.
- A *simplistic* description of an antenna and its radiation pattern will be developed. The electric and magnetic fields as a function of position and time will serve as the final description of classical electromagnetic — as it applies to the problem of this monograph.

> *The God said Let there be Light, and there was Light. The Light was made before ether sunne or moone was created therefore we must not attribute that to ye creatures that are Gods instruments.*
>
> — The Geneva Bible, 1560

During this year Einstein also held a full–time position at the Bern patent office, attended to his wife and small child and performed his physics research in his spare time. Since he was unable to obtain an academic position Einstein was isolated from the mainstream of the physics community — which may have attributed to his early successes. This position of isolation was repeated in his later life when he objected strongly to the underlying theories of quantum mechanics [Pais79], [Pais82], [Born71]. The *argument* with Neils Bohr was based on Einstein's contention that quantum mechanics as formulated at the time was not consistent with his principles of objectivity and causality that he found necessary for a *sensical* explanation of nature [Bohr49]. This argument has been popularized through Einstein's quote that *God does not play dice with the Universe.*

